Indian Institute of Information Technology Allahabad
Discrete Mathematical Structures (DMS)
Tentative marking scheme of C2 Review Test

Program: B.Tech. 2^{nd} Semester (IT) Full Marks: 20
Date: May 31, 2023 Time: 10:05 AM - 10:50 AM

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1. Check whether the following statements are true or false. Give a proper justification. [6]

   (a) If H is a commutative subgroup of a group G, then H is a normal subgroup of G.

   Solution: False. $H = \{I, (1,2)\}$ is a commutative subgroup of the symmetric group S_3, but H is not normal because $(13)H = \{(13), (123)\} \neq H(13) = \{(13), (132)\}$. [3]

   (b) Let $\mathbb{R}^+ = \{x \in \mathbb{R} \,|\, x > 0\}$. Let $(\mathbb{R}, +)$ and $(\mathbb{R}^+, .)$ be groups. Then the map $\phi : (\mathbb{R}, +) \to (\mathbb{R}^+, .)$ defined as $\phi(x) = e^x$ is a homomorphism.

   Solution: True. $\phi(x + y) = e^{x+y} = e^x e^y = \phi(x)\phi(y)$. [1]

   (c) $2\mathbb{Z} \cup 3\mathbb{Z}$ is a subring of $\mathbb{Z}$.

   Solution: False. Let $2, 3 \in 2\mathbb{Z} \cup 3\mathbb{Z}$. Then $2 + 3 = 5 \notin 2\mathbb{Z} \cup 3\mathbb{Z}$. [2]

2. Show that the complete bipartite graph $K_{3,3}$ is non-planar. [3]

   Solution: We know that for a connected and planar graph with no cycles of length three, we have $e \leq 2v - 4$. [1]

   Clearly, $K_{3,3}$ is connected and has no cycles of length three with $e = 9, v = 6$. This gives $9 \leq 2 * 6 - 4 = 8$, a contradiction. [2]

3. Show that if a simple graph G is isomorphic to its complement $\overline{G}$, then G has either $4k$ or $4k+1$ vertices for some natural number k. [3]

Solution: Since G is isomorphic to its complement $\overline{G}$, they have the same number of edges, i.e., $E(G) = E(\overline{G})$.
Note that $E(G) + E(\overline{G}) = \frac{n(n-1)}{2}$.
Then $E(G) = \frac{n(n-1)}{4}$. This is only possible if n or $n-1$ is divisible by 4.

4. Prove that among any three distinct integers, we can find two, say a and b, such that the number $a^3b - ab^3$ is a multiple of 10. [4]

Solution: Denote $E(a,b) = a^3b - ab^3 = ab(a-b)(a+b)$.Observe that if a and b are both odd or even then $a+b$ is even, and if one is odd and other is even then $a-b$ is even, it follows that $E(a,b)$ is always even. [1]

Hence we only have to prove that among any three integers, we can find two, a and b, with $E(a,b)$ divisible by 5. If one of the numbers is a multiple of 5, the property is true. [1]

If not, consider the pairs $\{1,4\}$ and $\{2,3\}$ of residues classes modulo 5. By the Pigeonhole Principle, the residues of two of the given numbers belong to the same pair. These will be a and b. If $a \equiv b(mod 5)$ then $a-b$ is divisible by 5, and so is $E(a,b)$. If not, then by the way we defined our pairs, $a+b$ is divisible by 5, and so again $E(a,b)$ is divisible by 5. [2]

5. Let S_3 denote the group of all permutations on set $X = \{1,2,3\}$. [4]

(a) Find the order of all the elements of S_3. [2]

Solution: We know that $S_3 := \{I, (12), (13), (23), (123), (132)\}$.
Here $o(I) = 1$, $o((12)) = 2$, $o((13)) = 2$, $o((23)) = 2$, $o((123)) = 3$ and $o((132)) = 3$.

(b) Write down all the (left) cosets of $H = \{I, (1,2)\}$ in S_3. [2]

Solution: $IH = (12)H = \{I, (1,2)\}$

$(13)H = (123)H = \{(1,3), (123)\}$, and

$(23)H = (132)H = \{(2,3), (132)\}$.